Counting edge-Kempe-equivalence classes for 3-edge-colored cubic graphs

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Abstract

Two edge colorings of a graph are *edge-Kempe equivalent* if one can be obtained from the other by a series of edge-Kempe switches. This work gives some results for the number of edge-Kempe equivalence classes for cubic graphs. In particular we show every 2-connected planar bipartite cubic graph has exactly one edge-Kempe equivalence class. Additionally, we exhibit infinite families of nonplanar bipartite cubic graphs with a range of numbers of edge-Kempe equivalence classes. Techniques are developed that will be useful for analyzing other classes of graphs as well.

1 Introduction and Summary

Back in the frosts of time, Alfred Bray Kempe introduced the notion of changing colorings by switching maximal two-color chains of vertices (for vertex colorings) [4] or edges (for edge colorings). The maximal two-color chains are now called *Kempe chains* and *edge-Kempe chains* respectively; switching the colors along such a chain is called a *Kempe switch* or *edge-Kempe switch* as appropriate. This process is of interest across the study of colorings. It is also of interest in statistical mechanics, where certain

dynamics in the antiferromagnetic q-state Potts model correspond to Kempe switches on vertex colorings [8], [9]. In some cases, these dynamics also correspond to edge-Kempe switches [7].

In the present work we are concerned with understanding when two edge-colorings are equivalent under a sequence of edge-Kempe switches and when not. We allow multiple edges on our (labeled) graphs; loops are prohibited (and will mostly be excluded by other constraints such as 3-edge colorability).

A single edge-Kempe switch is denoted by $-$. That is, if coloring c_i becomes coloring c_j after a single edge-Kempe switch, then $c_i - c_j$. If coloring c_j can be converted to coloring c_k by a sequence of edge-Kempe switches, then c_j and c_k are equivalent; we denote this by $c_j \sim c_k$. Because $\sim$ is an equivalence relation, we may consider the equivalence classes on the set of colorings of a graph G edge-colored with n colors. In this paper we focus on the *number* of edge-Kempe equivalence classes and denote this quantity by $K'(G,n)$. (In other work this has been denoted $\mathrm{Ke}(L(G),n)$ [6] and $\kappa_E(G,n)$ [5].)

Note that any global permutation of colors can be achieved by edge-Kempe switches because the symmetric group S_n is generated by transpositions. Thus two colorings that differ only by a permutation of colors are edge-Kempe equivalent.

Recall that $\Delta(G)$ is the largest vertex degree in G and that $\chi'(G)$ is the smallest number of colors needed to properly edge-color G. When more colors are used than possibly needed to edge-color the graph, then there is but a single edge-Kempe equivalence class, i.e., when $n > \chi'(G)+1$ then $K'(G,n) = 1$ [6, Thm. 3.1]. More is known if $\Delta(G)$ is restricted; when $\Delta(G) \leq 4, K'(G,\Delta(G)+2) = 1$ [5, Thm. 2] and when $\Delta(G) \leq 3, K'(G,\Delta(G)+1) = 1$ [5, Thm. 3]. For bipartite graphs there is a stronger result: when $n > \Delta(G)$, $K'(G,n) = 1$ [6, Thm. 3.3]. Little is known about $K'(G,\Delta(G))$.

This paper focuses on cubic graphs, particularly those that are 3-edge colorable. Mohar suggested classifying cubic bipartite graphs with $K'(G,3) = 1$ [6]; we provide a partial answer here. Mohar also points out in [6] that it follows from a result of Fisk in [1] that every planar 3-connected cubic bipartite graph G has $K'(G,3) = 1$. We show (in Section 4) that for G planar, bipartite, and cubic, G has $K'(G,3) = 1$.

The remainder of the paper proceeds as follows. Section 2 introduces decompositions of cubic graphs along 2- or 3-edge cuts that preserve planarity and bipartiteness. The theorems in Section 3 use the edge-cut decompositions

to combine and decompose 3-edge colorings. We also show that any edge-Kempe equivalence can avoid color changes at a particular vertex. Then, in Section 4 we compute $K'(G,3)$ in terms of the edge-cut decomposition of G, and exhibit infinite families of simple nonplanar bipartite cubic graphs with a range of numbers of edge-Kempe equivalence classes.

2 Decompositions of Cubic Graphs

Any 3-edge cut of a cubic graph may be used to decompose a cubic graph G into two cubic graphs G_1, G_2 as follows. For 3-edge cut $E_C = \{(s_{11}s_{21}), (s_{12}s_{22}), (s_{13}s_{23})\}$ where vertices s_{1j} are on one side of the cut and s_{2j} on the other, let the induced subgraphs of $G \setminus E_C$ separated by E_C be G'_1, G'_2. Then for $i = 1, 2$ define G_i by $V(G_i) = V(G'_i) \cup v_i$ and $E(G_i) = E(G'_i) \cup E_{C_i}$ where $E_{C_i} = \{(v_i s_{ij}) |\ j = 1, 2, 3\}$, as is shown in Figure 1. This decomposition will be written as $G = G_1 \curlyvee G_2$.

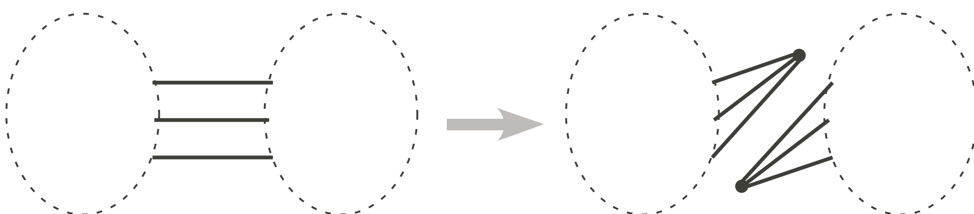

Figure 1: Decomposing a graph over a 3-edge cut.

A similar decomposition is defined analogously for a 2-edge cut of a cubic graph. Here G has 2-edge cut $E_C = \{(s_{11}s_{21}), (s_{12}s_{22})\}$ and for $i = 1, 2$ we define G_i by $V(G_i) = V(G'_i)$ and $E(G_i) = E(G'_i) \cup e_i$ where $e_i = \{(s_{i1}s_{i2})\}$. This decomposition will be written as $G = G_1 \mp G_2$.

For both of these decompositions, we say the edge cut is nontrivial if both G_1 and G_2 have fewer vertices than G. Using nontrivial edge cuts, we may decompose a cubic graph G into a set of smaller graphs $\{G_i\}$ where each G_i has no nontrivial edge cuts (but may have additional multiple edges).

Notice that these decompositions are reversible, though not uniquely so. Consider two cubic graphs G_1, G_2. Form $G_1 \curlyvee G_2$ by distinguishing a vertex on each (v_1, v_2 respectively) and identifying the edges incident to v_1 with the edges incident to v_2. *A priori*, there are many ways to choose v_1, v_2 and many

ways to identify their incident edges. We will abuse the notation $G_1 \Yup G_2$ by using it to denote a particular one of these many choices. Similarly, $G_1 \mathbin{\mathtt{I}} G_2$ can be formed by choosing an edge $e_i = (s_{i1}s_{i2})$ from each G_i, deleting e_i, and then adding the edges $\{(s_{11}s_{21}), (s_{12}s_{22})\}$. Note that constructing $G_1 \mathbin{\mathtt{I}} G_2$ is equivalent to cutting an edge of G_2 and inserting it into a single edge of G_1.

Lemma 2.1. *Let G be a cubic graph. If $G = G_1 \Yup G_2$ or $G = G_1 \mathbin{\mathtt{I}} G_2$, then G is planar if and only if G_1 and G_2 are planar.*

Proof. Suppose that G has a cellular embedding on the sphere. Then the removal of an edge cut E_C separates G into two subgraphs, G_1', G_2' embedded on the sphere, each of which is contained in one of two disjoint discs D_1, D_2. Note that the resulting degree-1 and degree-2 vertices of each subgraph are on its outer face (relative to D_i) as in Figure 2. If E_C was a 2-edge cut, edges

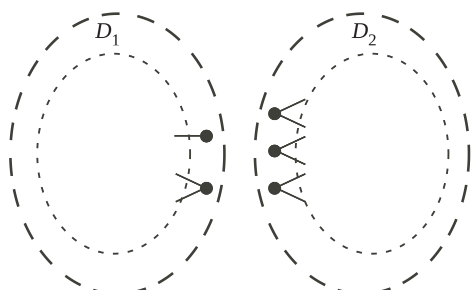

Figure 2: A sample configuration of planar G_1', G_2'.

may be added on the outside face that join these vertices to create planar G_i. If E_C was a 3-edge cut, add vertices v_1, v_2 on the outside faces of discs D_1, D_2 respectively, and join v_i to the degree-1 and degree-2 vertices in D_i to create planar G_i.

Conversely, spherical embeddings of G_1 and G_2 may be converted to planar drawings with distinguished vertices v_1, v_2 or edges e_1, e_2 on the outside faces of discs D_1, D_2 respectively. Removing v_1, v_2 (resp. e_1, e_2) produces G with three edges (resp. two edges) of a cut missing. Any desired pairing of the vertices may be completed on a sphere without edges crossing by using judicious placement of D_i (and perhaps flipping one over). This will result in $G_1 \Yup G_2$ (resp. $G_1 \mathbin{\mathtt{I}} G_2$).

$\square$

Lemma 2.2. *Let G be a cubic graph. If $G = G_1 \mathbin{\mathrm{I}} G_2$, or $G = G_1 \mathbin{\mathrm{Y}} G_2$, then G is bipartite if and only if G_1 and G_2 are bipartite.*

Proof. If G is a cubic bipartite graph with nontrivial 2-edge cut, then let there be m_j vertices from part j on side 1; if both cut edges emanate from part 1 then $3m_1 - 2 = 3m_2$ which is impossible. Thus each cut edge must emanate from a different part on side i of the cut, so both removing the edge cut and placing edges on each side maintains bipartition.

Suppose G is a bipartite cubic graph with nontrivial 3-edge cut E_C and G_1', G_2' the induced subgraphs of $G \setminus E_C$. For a bipartition of G to descend naturally to bipartitions of G_1, G_2, the edges of E_C must be incident only to vertices in G_i' that are in the same part of G. Therefore, assume this is not the case and (without loss of generality) that two of the edges of E_C are incident to one part of G_1' and the remaining edge of E_C is incident to the other part of G_1'. Let G_1' have m_j vertices belonging to part j of G. There are $3m_1 - 1$ edges emanating from part 1 of G_1' that must be incident to vertices of part 2 of G_1'. On the other hand, there are $3m_2 - 2$ edges emanating from part 2 of G_1' that must be incident to vertices in part 1. Thus $3m_1 - 1 = 3m_2 - 2$, which is impossible.

Conversely, if G_1, G_2 are bipartite, with distinguished $e_1 = s_{11}s_{12}, e_2 = s_{21}s_{22}$ for the purpose of forming $G_1 \mathbin{\mathrm{I}} G_2$, then the bipartition of G_1 extends to $G_1 \mathbin{\mathrm{I}} G_2$ by assigning s_{12} (resp. s_{22}) to the opposite part as s_{11} (resp. s_{21}). Similarly, if G_1, G_2 are bipartite, with distinguished v_1, v_2 for the purpose of forming $G_1 \mathbin{\mathrm{Y}} G_2$, then use the bipartition of G_1 and assign v_2 to the opposite part as v_1 to induce a bipartition of $G_1 \mathbin{\mathrm{Y}} G_2$. □

Theorem 2.3. *A cubic graph H that is 2-connected but not 3-connected may be decomposed via $\mathbin{\mathrm{I}}$ into a set of cubic loopless graphs $\{H_i\}$ where each H_i is 3-connected.*

Proof. The proof is inductive on the number of vertices of H. Because H is 2-connected but not 3-connected, there exists a 2-vertex separating set. Figure 3 shows the three possible edge configurations for a 2-vertex separating set of a cubic graph, along with (at top) associated 2-edge cuts. Each 2-edge cut can be used to form $H = H_1 \mathbin{\mathrm{I}} H_2$, and $|H_j| < |H|$ so the inductive hypothesis holds for H_j.

□

It is worth noting that while the decomposition can create multiple edges, any multiple edge in a cubic graph will be associated with a 2-edge cut. Thus

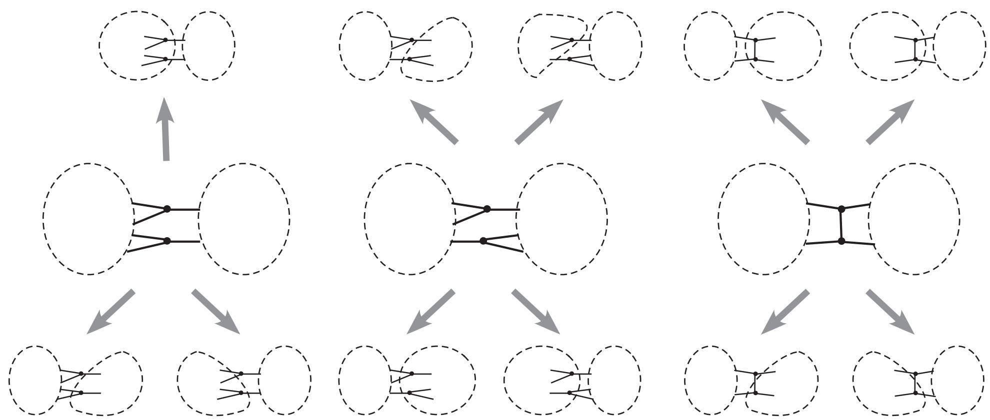

Figure 3: 2-vertex separating sets with associated 2-edge cuts (top) and 3-edge cuts (bottom).

the final set of H_j will be composed of theta graphs, and graphs with no multiple edges.

Corollary 2.4. *The I decomposition of 2-connected cubic graphs given by Theorem 2.3 preserves both planarity and bipartiteness.*

Proof. This follows from Lemmas 2.1 and 2.2. $\square$

An alternative decomposition using the Y product can also be found. This is because every 2 vertex separating set is also associated with a 3-edge cut as seen in Figure 3(bottom). This decomposition also preserves planarity and bipartiteness.

3 Manipulating and Composing Colorings

We begin by showing that we can fix the colors on the edges incident to a given vertex, and accomplish any sequence of edge-Kempe switches without changing the fixed colors. As a result, representatives of all edge-Kempe equivalence classes will be present in the set of colorings with fixed colors at a vertex. The following theorem holds for all base graphs G, not just cubic graphs, and all $n \geq \chi'(G)$.

Theorem 3.1. *If $c \sim d$ are two proper edge colorings of a loopless graph G, and there exists a vertex v such that $c(e_i) = d(e_i)$ for all e_i incident to v,*

then there exists a sequence of edge-Kempe switches from c to d that never change the colors on the edges incident to v.

Recall that $o_i - o_{i+1}$ is the notation for two colorings that differ by exactly one edge-Kempe switch. It will be useful to have a further notation for the switch itself. Let $s_i = (\{p_{i_1}, p_{i_2}\}, t_i)$ where $\{p_{i_1}, p_{i_2}\}$ is the pair of colors to be switched on the chain t_i of G. Then write $o_i -_{s_i} o_{i+1}$, if o_{i+1} is obtained from o_i by switching colors $\{p_{i_1}, p_{i_2}\}$ on chain t_i. Considering S_n as acting on the set of colors $\{1, \ldots, n\}$, let $\pi_i \in S_n$ be the transposition $\pi_i(p_{i_1}) = p_{i_2}, \pi_i(p_{i_2}) = p_{i_1}$.

The idea of the proof is as follows. Each time a switch $s_i = (\{p_{i_1}, p_{i_2}\}, t_i)$ affects an edge incident to v, replace it by making all other $\{p_{i_1}, p_{i_2}\}$ switches in the graph. This results in a coloring of the graph that is equivalent to the original, at the same stage, via a global color permutation. Therefore we need to track the colors to be switched on t_k, for $k > i$. Each switch s_k that does not affect an edge incident to vertex v will be replaced by a switch, on the same chain t_k, of the colors that are currently on that chain. Our proof gives this precisely as an algorithm.

Proof. Suppose that $c = o_0 -_{s_0} o_1 -_{s_1} \cdots -_{s_{n-1}} o_m = d$, and there is at least one i such that $v \in t_i$. Let σ_0 be the identity permutation. For $0 \leq i \leq m-1$, replace s_i with a set of edge-Kempe switches $\hat{s}_i$ as follows. Set $\hat{\pi}_i = \sigma_i \pi_i \sigma_i^{-1}$ so that $\hat{\pi}_i(\sigma_i(p_{i_1})) = \sigma_i(p_{i_2})$.
If $v \notin t_i$ then set $\hat{s}_i = \{(\{p_{i_1}, p_{i_2}\}, t_i)\}$ and $\sigma_{i+1} = \sigma_i$.
If $v \in t_i$ then for $\{t_j\}$ the edge-Kempe chains of o_i in colors $\{p_{i_1}, p_{i_2}\}$, set $\hat{s}_i = \{(\{\sigma_i(p_{i_1}), \sigma_i(p_{i_2})\}, t_j) |\ t_j \neq t_i\}$ and $\sigma_{i+1} = \sigma_i \pi_i$. Note that the set $\hat{s}_i$ may be empty if t_i is the only $\{p_{i_1}, p_{i_2}\}$ chain in o_i.

Define $\hat{o}_{i+1}$ to be the result of performing the sets of switches $\hat{s}_1, \ldots, \hat{s}_i$ to c. We show that $\hat{o}_{i+1}$ and o_i are equivalent up to a global color permutation by σ_i. Recall that $o_i(e)$ is the color assigned to edge e by o_i. We must show that on each edge e, $\hat{o}_{i+1}(e) = \sigma_{i+1} o_{i+1}(e)$. We proceed by induction and so assume that for $k \leq i$, $\hat{o}_k(e) = \sigma_k o_k(e)$.

There are 5 cases.
First suppose $v \notin t_i$.

Case 1a. If $e \in t_i$ then $\hat{o}_{i+1}(e) = \hat{\pi}_i \hat{o}_i(e)$ because $\hat{\pi}_i$ is the action of switch $\hat{s}_i$. By definition of $\hat{\pi}_i$ and using the inductive hypothesis for $\hat{o}_i$, $\hat{\pi}_i \hat{o}_i(e) = (\sigma_i \pi_i \sigma_i^{-1})(\sigma_i o_i(e))$. Simplifying, we have $\sigma_i \pi_i o_i(e) = \sigma_i o_{i+1}(e)$ (by action of s_i on o_i), which, by definition of σ_{i+1} in this case, equals $\sigma_{i+1} o_{i+1}(e)$

as desired. Similar reasoning justifies the remaining cases so we present them in an abbreviated fashion.

Case 1b. If $e \notin t_i$ then $\hat{o}_{i+1}(e) = \hat{o}_i(e) = \sigma_i o_i(e) = \sigma_{i+1} o_{i+1}(e)$.

Now suppose $v \in t_i$.

Case 2a. If $o_i(e) \notin \{p_{i_1}, p_{i_2}\}$ then $\hat{o}_{i+1}(e) = \hat{o}_i(e) = \sigma_i o_i(e) = (\sigma_i \pi_i) o_i(e) = \sigma_{i+1} o_{i+1}(e)$.

Case 2b. If $o_i(e) \in \{p_{i_1}, p_{i_2}\}$ and $e \in t_i$, then the color on e does not change from $\hat{o}_i$ to $\hat{o}_{i+1}$ while it did change from o_i to o_{i+1}. Thus, $\hat{o}_{i+1}(e) = \hat{o}_i(e) = \sigma_i o_i(e) = \sigma_i \pi_i \pi_i o_i(e) = \sigma_{i+1} o_{i+1}(e)$.

Case 2c. If $o_i(e) \in \{p_{i_1}, p_{i_2}\}$ and $e \notin t_i$, then the color on e does change from $\hat{o}_i$ to $\hat{o}_{i+1}$ while it did not change from o_i to o_{i+1}. Thus, $\hat{o}_{i+1}(e) = \hat{\pi}_i \hat{o}_i(e) = (\sigma_i \pi_i \sigma_i^{-1})(\sigma_i o_i(e)) = (\sigma_i \pi_i) o_i(e) = \sigma_{i+1} o_{i+1}(e)$.

Finally, we consider $\hat{o}_m$ and compare it to d. Note c and d have the same colors on v by hypothesis, and the total number of colors used in d is n. If $n \leq \deg(v)+1$, then at most one color is not represented at v and σ_m must be the identity permutation; thus $\hat{o}_m = o_m = d$. If $n > \deg(v)+1$, then it is possible that some colors that do not occur at v are globally permuted between o_m and $\hat{o}_n$. In this case, additional edge-Kempe switches that globally permute colors can be applied to $\hat{o}_m$ so that the coloring now matches d.

□

This result shows when counting the number of edge-Kempe equivalence classes it is sufficient to consider only colorings of G that are different up to global color permutation. To make this observation precise requires careful definition of an *edge-Kempe-equivalence graph* of a graph. This will be done in [2].

Returning to cubic graphs, we next consider how combining graphs affects $K'(G, n)$. Let G_1, G_2 be two 3-edge-colorable cubic graphs and distinguish a vertex on each (v_1, v_2) for the purpose of forming $G_1 \curlyvee G_2$. Recall that in addition to the choice of v_1, v_2, there are multiple ways their incident edges may be identified; by $G_1 \curlyvee G_2$ we mean some particular set of these choices. Let $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ be the ordered sets of edges in G_1 and G_2 that will be identified in $G_1 \curlyvee G_2$. Similarly, choose a distinguished edge in each graph $(x \in G_1, y \in G_2)$ for the purpose of forming $G_1 \mp G_2$. The following several results relate 3-edge colorings of G_1 and G_2 to those of $G_1 \curlyvee G_2$ and $G_1 \mp G_2$.

Definition 3.2. Let c, d be proper edge colorings of G_1, G_2 respectively. There exists a proper coloring $\hat{d}$ of G_2 such that $c(x_i) = \hat{d}(y_i)$ for $i = 1, 2, 3$, and such that $d, \hat{d}$ are the same up to a permutation of the colors ($d \sim \hat{d}$). Define $(c \curlyvee d)$ to be the proper coloring of $G_1 \curlyvee G_2$ given by

$$(c \curlyvee d)(e) = \begin{cases} c(e) & \text{if } e \in G_1 \\ \hat{d}(e) & \text{if } e \in G_2 \\ c(e) = \hat{d}(e) & \text{if } e \text{ is the edge resulting from identifying } x_i \text{ and } y_i. \end{cases}$$

Similarly, there exists a proper coloring $\tilde{d}$ of G_2 such that $c(x) = \tilde{d}(y)$ and such that $d, \tilde{d}$ are the same up to a global permutation of the colors. Define $(c \mp d)$ to be the proper coloring of $G_1 \mp G_2$ given by

$$(c \mp d)(e) = \begin{cases} c(e) & \text{if } e \in G_1 \\ \tilde{d}(e) & \text{if } e \in G_2 \\ c(e) = \tilde{d}(e) & \text{if } e \text{ is one of the edges added after deleting } x \text{ and } y. \end{cases}$$

Two cases of the Parity Lemma ([3]) will be useful.

Lemma 3.3. *Let E_C be an edge cut of a of a 3-edge-colorable cubic graph G and c be any proper 3-edge coloring of G. Then*
(a) if E_C is a 2-edge cut, then $c(E_C)$ uses exactly one color, and
(b) if E_C is a 3-edge cut, then $c(E_C)$ uses all three colors.

Theorem 3.4. *Every 3-edge coloring f of $G = G_1 \curlyvee G_2$ (resp. $G = G_1 \mp G_2$) can be written as $c_1 \curlyvee d_1$ (resp. $c_1 \mp d_1$) where c_1 is some 3-edge coloring of G_1 and d_1 is some 3-edge coloring of G_2.*

Proof. Consider a 3-edge coloring f of $G = G_1 \curlyvee G_2$. There is a 3-edge cut E_C corresponding to the decomposition $G_1 \curlyvee G_2$. By Lemma 3.3(b), each $e_i \in E_C$ must be a different color in c. Therefore considering f on the edges of G_1 (and particularly at v_1), it is still a proper coloring c_1, and likewise f considered on G_2 is a proper coloring d_1. The result for $\mp$ is similarly an immediate corollary of Lemma 3.3.

□

Implicit in the preceding results is the following.

Corollary 3.5. *If $G = G_1 \curlyvee G_2$ or $G = G_1 \mp G_2$, then G is 3-edge colorable if and only if G_1 and G_2 are 3-edge colorable.*

Next we note how edge-Kempe equivalences on the colorings of G_1 and G_2 transfer to edge-Kempe equivalences in combinations of these graphs.

Lemma 3.6. *Let 3-edge colorings $c_1 \sim c_2$ in G_1 and $d_1 \sim d_2$ in G_2. Then $(c_1 \curlyvee d_1) \sim (c_2 \curlyvee d_2)$ in $G_1 \curlyvee G_2$ and $(c_1 \mp d_1) \sim (c_2 \mp d_2)$ in $G_1 \mp G_2$.*

Proof. Using the notation from Definition 3.2, let $c_2' \sim c_2$ by global color permutation such that $c_2'(x_i) = c_1(x_i)$ for $i = 1, 2, 3$. By Theorem 3.1, there exists a sequence of edge-Kempe switches in G_1 that exhibits $c_1 \sim c_2'$ and that never changes the color of any edge incident to v_1. Similarly, define $\hat{d}_2' \sim \hat{d}_2 \sim d_2$ such that there is a sequence of edge-Kempe switches in G_2 that exhibits $\hat{d}_1 \sim \hat{d}_2'$ and that never changes the color of any edge incident to v_2. Then $(c_1 \curlyvee d_1) = (c_1 \curlyvee \hat{d}_1) \sim (c_2' \curlyvee \hat{d}_1) \sim (c_2' \curlyvee \hat{d}_2') \sim (c_2 \curlyvee \hat{d}_2) = (c_2 \curlyvee d_2)$.

For the $\mp$ composition, assume without loss of generality that $c_1(x) = d_1(y)$. Let $c_2'' \sim c_2$ by global color permutation such that $c_2''(x) = c_1(x)$ and $d_2'' \sim d_2$ by global color permutation such that $d_2''(y) = d_1(y)$. By Lemma 3.3, the two edges created after deleting x, y will be assigned the same color in any proper 3-coloring of $G_1 \mp G_2$, so fixing the color on one will also fix the color on the other. Hence, $(c_1 \mp d_1) \sim (c_2'' \mp d_1) \sim (c_2'' \mp d_2'') \sim (c_2 \mp d_2)$. □

Lemma 3.7. *Let G_1, G_2 be 3-edge colorable cubic graphs with $G_1 \curlyvee G_2$ and $G_1 \mp G_2$ particular compositions of the two. If $(c_1 \curlyvee d_1) \sim (c_2 \curlyvee d_2)$ in $G_1 \curlyvee G_2$ (resp. $(c_1 \mp d_1) \sim (c_2 \mp d_2)$ in $G_1 \mp G_2$) then $c_1 \sim c_2$ in G_1 and $d_1 \sim d_2$ in G_2.*

Proof. It is sufficient to show this when $(c_1 \curlyvee d_1) -_s (c_2 \curlyvee d_2)$ and $(c_1 \mp d_1) -_s (c_2 \mp d_2)$, where $s = (p, t)$ with p a pair of colors and t an edge-Kempe chain. If $t \subset G_1$ or $t \subset G_2$, then the lemma holds. Otherwise, $t \cap E_C \neq \emptyset$, and t must use exactly 2 edges of E_C because every edge-Kempe chain of a proper 3-edge coloring of a cubic graph is a cycle. The decomposition $G_1 \curlyvee G_2$ (resp. $G_1 \mp G_2$) over E_C will decompose t into an edge-Kempe chain t_1 of G_1 and t_2 of G_2. Then $c_1 -_{(p,t_1)} c_2$ in G_1 and $d_1 -_{(p,t_2)} d_2$ in G_2. □

Theorem 3.8. *Let G_1, G_2 be cubic graphs. If $K'(G_1, 3) = a$ and $K'(G_2, 3) = b$, then $K'(G_1 \curlyvee G_2, 3) = K'(G_1 \mp G_2, 3) = ab$.*

Proof. Choose colorings $c_1, \ldots, c_a$, one from each of the a edge-Kempe-equivalence classes of G_1, and likewise choose colorings $d_1, \ldots, d_b$, one from each of the b edge-Kempe-equivalence classes of G_2. Every 3-edge coloring f of $G_1 \curlyvee G_2$ can be written as $f = \hat{c} \curlyvee \hat{d}$ by Theorem 3.4. $\hat{c} \sim c_i$ for some $c_i \in \{c_1, \ldots, c_a\}$, and $\hat{d} \sim d_j$ for some $d_j \in \{d_1, \ldots, d_b\}$, so by Lemma 3.6 $f \sim c_i \curlyvee d_j$ for some $c_i \in \{c_1, \ldots, c_a\}, d_j \in \{d_1, \ldots, d_b\}$. Further by Lemma 3.7, $c_{i_1} \curlyvee d_{j_1} \sim c_{i_2} \curlyvee d_{j_2}$ only when $i_1 = i_2, j_1 = j_2$. Therefore there are ab edge-Kempe-equivalence classes of $G_1 \curlyvee G_2$. The proof for $G_1 \mp G_2$ is identical. □

4 Results on $K'(G, 3)$

Theorem 3.8 can be extended to compose several graphs, or alternatively to decompose a graph into many smaller pieces. We will use the theorem below in both contexts to get results about possible numbers of edge-Kempe equivalence classes for cubic graphs.

Theorem 4.1. *Let G be a 3-edge colorable cubic graph. Then $K'(G, 3) = \prod_i K'(G_i, 3)$ where $\{G_i\}$ is a decomposition of G along nontrivial 2-edge cuts or 3-edge cuts.*

Proof. This follows from multiple applications of Theorem 3.8. $\square$

4.1 Planar, cubic, bipartite graphs

The following theorem answers a question from [6, Section 3].

Theorem 4.2. *Let H be a 2-connected, but not 3-connected, planar bipartite cubic graph. Then $K'(H, 3) = 1$.*

Proof. By Theorem 2.3, H may be decomposed into $\{H_i\}$ where all H_i are 3-connected. By Lemmas 2.1 and 2.2, all H_i are planar and bipartite. As pointed out in [6], it follows from [1] that all 3-connected planar bipartite cubic graphs G have $K'(G, 3) = 1$ so for all H_i, $K'(H_i, 3) = 1$. It then follows from Theorem 4.1 that $K'(H, 3) = 1$. $\square$

Recall that if G is cubic and bipartite then it must be bridgeless. Thus we get the following result.

Corollary 4.3. *Let H be a planar bipartite cubic graph. Then $K'(H, 3) = 1$.*

4.2 Nonplanar, cubic, bipartite graphs

Matters are quite different for *nonplanar* bipartite cubic graphs. It is well known that $K_{3,3}$ has two different edge-colorings (shown in Figure 4). In each of these colorings, each color-pair forms a Hamilton cycle. Therefore, any edge-Kempe switch results in a permutation of the colors and neither coloring of Figure 4 can be obtained from the other. Thus, there are two edge-Kempe equivalence classes, i.e. $K'(K_{3,3}, 3) = 2$.

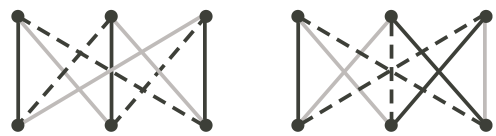

Figure 4: The two colorings of $K_{3,3}$.

Lemma 4.4. *Every simple bipartite nonplanar cubic graph B with $n \leq 10$ has $K'(B,3) > 1$.*

Proof. Every simple bipartite nonplanar cubic graph is a subdivision of $K_{3,3}$. To maintain the bipartition and avoid multiple edges, $K_{3,3}$ must be subdivided with at least 4 vertices, two on each of two edges. These edges may be independent or may be incident.

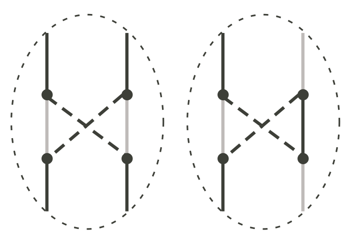

Figure 5: The two possible colorings around subdivided independent or incident edges.

Any coloring of the original graph extends to either one or two new (edge-Kempe equivalent) colorings, as is shown in Figure 5. If a coloring had three Hamilton cycles before subdivision (as is true for both colorings of $K_{3,3}$), at most it gains an isolated edge-Kempe cycle after subdivision of this sort. Thus when subdividing $K_{3,3}$ with a single 4-vertex subdivision, there still exist two colorings that are not edge-Kempe-equivalent. □

Further examples of nonplanar cubic bipartite graphs with $K'(G,3) > 1$ will be given in Section 4.3. In contrast, Figure 6 shows a bipartite nonplanar cubic graph U with 12 vertices and $K'(U,3) = 1$. $K'(U,3)$ was computed manually and verified using custom *Mathematica* code. We can use U to produce an interesting infinite class of graphs.

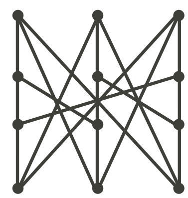

Figure 6: A nonplanar bipartite cubic graph that has a single edge-Kempe equivalence class.

Theorem 4.5. *There exists an infinite family of simple nonplanar 3-connected bipartite cubic graphs* U_k *with* $2 + 10k$ *vertices and* $K'(U_k, 3) = 1$.

Proof. Let $U_k = U \curlyvee \cdots$ (k copies) $\ldots \curlyvee U$. By Theorem 3.8, $K'(U_k, 3) = 1$. Graphs U_2, U_3, and U_4 are shown in Figure 7. $\square$

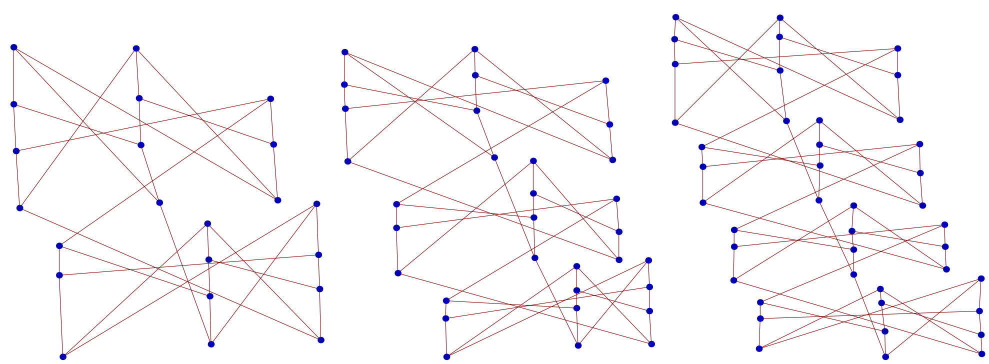

Figure 7: Three members of an infinite family of bipartite nonplanar cubic graphs U_k, each member of which has a single edge-Kempe equivalence class.

By $\curlyvee$ composition of U with a planar cubic bipartite graph with $n - 10$ vertices we get the following more general result.

Theorem 4.6. *For any* $n \geq 18$ *there is a simple, nonplanar, bipartite, 3-connected, cubic graph* G *with* n *vertices and* $K'(G, 3) = 1$.

Notice that similar results can be obtained for graphs that are only 2-connected as well by using the $\mathrel{\text{\textasciitilde}}$ composition.

4.3 Cubic graphs with $K'(G,3) > 1$

We can form $K_{3,3} \curlyvee G$ with any 3-connected cubic graph G to obtain a 3-connected nonplanar cubic graph. By Theorem 3.8,

$$K'(K_{3,3} \curlyvee G, 3) = K'(K_{3,3}, 3)K'(G, 3) = 2K'(G, 3).$$

Theorem 4.7. *For every even $n \geq 8$, there exists a 3-connected nonplanar cubic graph G with n vertices and exactly 2 edge-Kempe equivalence classes.*

Proof. Form $K_{3,3} \curlyvee G$ with any 3-connected planar cubic graph G on $n - 4$ vertices to obtain a 3-connected nonplanar cubic graph with n vertices and $K'(K_{3,3} \curlyvee G, 3) = 2$. □

Corollary 4.8. *For every even $n \geq 12$, there exists a 3-connected nonplanar bipartite cubic graph G with n vertices and exactly 2 edge-Kempe equivalence classes.*

Proof. Form $K_{3,3} \curlyvee G$ with any 3-connected planar cubic bipartite graph G on $n - 4$ vertices. The smallest 3-connected planar cubic bipartite graph has 8 vertices. □

More generally, once we have one example with k edge-Kempe equivalence classes then there will be an infinite family of them with the same number of classes.

Theorem 4.9. *If $\hat{G}$ is a cubic graph on $\hat{n}$ vertices with k edge-Kempe equivalence classes then for every even $n \geq \hat{n} + 6$, there exists a cubic graph on n vertices with exactly k edge-Kempe equivalence classes. Further, if $\hat{G}$ is planar then a planar family exists, if $\hat{G}$ is bipartite then a bipartite family exists and if $\hat{G}$ is 3-connected then a 3-connected family exists.*

Proof. Compose $\hat{G}$ with any cubic planar bipartite graph on $n+2-\hat{n}$ vertices using the $\curlyvee$ operation. The result follows from Theorem 3.8. □

We can make graphs with increasingly large numbers of edge-Kempe equivalence classes this way as well.

Theorem 4.10. *For every $k \geq 1$, there exists a 3-connected nonplanar bipartite cubic graph G with $4k + 2$ vertices and 2^k edge-Kempe equivalence classes.*

Proof. For $k \geq 1$, take $K_{3,3} \Yup \cdots$ (k copies) $\ldots \Yup K_{3,3}$, which has $2 + 4k$ vertices. By Theorem 3.8, it has 2^k edge-Kempe equivalence classes. This produces the desired graph.

$\square$

Theorem 4.11. *For every simple nonplanar (bipartite) cubic graph G with n vertices, there exists an infinite family of nonplanar (bipartite) cubic graphs G_k such that G_k has $6k + n$ vertices and $2^k K'(G, 3)$ edge-Kempe equivalence classes.*

Proof. Take $G \mathbin{\mathtt{I}} K_{3,3} \mathbin{\mathtt{I}} \ldots \mathbin{\mathtt{I}} K_{3,3}$. $\square$

5 Computations of $K'(G, 3)$

Computing $K'(G, 3)$ for particular G, or for families of graphs, is surprisingly difficult. A single computation can be done by brute force by computer, but constructing a proof is another matter. As examples of the kinds of arguments needed to determine $K'(G, 3)$, we analyze Möbius ladder graphs, prism graphs, and crossed prism graphs.

Theorem 5.1. *Let ML_k be the Möbius ladder graph on $2k$ vertices, let Pr_k be the prism graph on $2k$ vertices, and let CPr_k be the crossed prism graph on $4k$ vertices.*

1. *$K'(ML_k, 3) = 1$ when k is even and $K'(ML_k, 3) = 2$ when k is odd.*
2. *$K'(Pr_k, 3) = 1$.*
3. *$K'(CPr_k, 3) = 1$.*

Note that Pr_k is planar, and bipartite exactly when k is even; ML_k is toroidal.

Proof. Our arguments are inductive.

First, consider the edge coloring of ML_k given at left in Figure 8, and note that it only exists for k odd. Every edge-Kempe chain in this coloring is a Hamilton circuit, so this coloring represents a edge-Kempe-equivalence class of of ML_k. Now consider any other 3-edge coloring of ML_k. If it has a square colored as shown at right in Figure 8, then the square may be removed (and the remaining half-edges glued together) to produce a 3-edge coloring

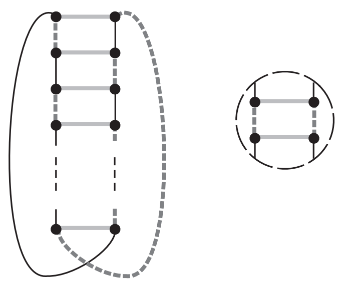

Figure 8: A tri-Hamiltonian edge coloring of ML_k for k odd (left) with a square from some other colorings of ML_k (right).

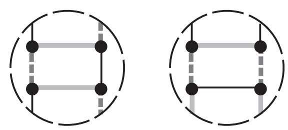

Figure 9: Colorings of squares from ML_k that are edge-Kempe-equivalent to a removable colored square of ML_k.

of ML_{k-2}. If there is no such square in the coloring, then every square must be colored as one of the options shown in Figure 9. In either case, we can do a single edge-Kempe switch to produce an edge-Kempe-equivalent coloring that contains a removable square. Therefore $K'(ML_k, 3) = K'(ML_{k-2}, 3)$. To complete the proof, it suffices to show (which direct computation does) that $K'(ML_4, 3) = 1$ and $K'(ML_3, 3) = 2$.

Next consider any 3-edge coloring of Pr_k. The same argument as for ML_k applies, so by removing a square we see that $K'(Pr_k, 3) = K'(Pr_{k-2}, 3)$. Because $K'(Pr_3, 3) = K'(Pr_4, 3) = 1$ by direct computation, it then follows that $K'(Pr_k, 3) = 1$.

Finally, consider any 3-edge coloring of CPr_k. Any crossed square must have one of the local colorings shown in Figure 10. For the leftmost two

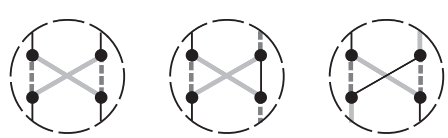

Figure 10: The possible colorings of a crossed square of CPr_k.

colorings of Figure 10, the crossed square may be removed (and the remaining half-edges glued together) to produce a 3-edge coloring of CPr_{k-1}. If there are only crossed squares with coloring type of the rightmost coloring in Figure 10, we can do a single edge-Kempe switch to produce an edge-Kempe-equivalent coloring that contains a removable crossed square. (A parity argument shows that there must be at least two edge-Kempe chains in a relevant color pair.) Because $K'(CPr_2, 3) = 1$ by direct computation, it then follows that $K'(CPr_k, 3) = 1$.

□

6 Areas for future work

Two major questions remain about $K'(G)$ for cubic, nonplanar, bipartite graphs. First, while we have shown that there are nonplanar cubic bipartite graphs with $K'(G, 3) = 1$ and also some with $K'(G, 3) > 1$, there is as yet no characterization for when each is true. Second, using *Mathematica* we have found bipartite cubic graphs where $K'(G, 3) = 1, 2, 3, 4, 6, 8, 9, 15, 17, 35, 131$. Which natural numbers, and in particular which primes, k are achievable as $K'(G, 3) = k$ for G a cubic nonplanar bipartite 3-connected graph, with no nontrivial edge cuts? These same questions can be asked for cubic 3-colorable graphs more generally: which have $K'(G, 3) = 1$, and what possible $K'(G, 3)$ values can occur?

Beyond just examining the number of edge-Kempe connected components, what is the structure of the edge-Kempe-equivalence Graph of G, whose vertices represent colorings of G and whose edges represent single edge-Kempe switches? This is the topic of [2].

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